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LETTER TO THE EDITOR

On large-order perturbation calculus for anharmonic potentials

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Abstract. We rederive the formula determining the large-order coefficients in the perturbation expansion of the ground state energy for the general anharmonic potential. We show that the transition from the Borel summability to the Borel non-summability of the series, connected with the change of the symmetry of the potential and with the occurrence of the degenerate minima, may be described by a discontinuously varying exponent.

The first analysis of the large behaviour of the perturbation calculus for the zero space dimension anharmonic oscillator was made by Bender and Wu (1973) who showed using semiclassical WKB methods that the series are divergent (in the usual sense). Then it was shown that useful information could be extracted even from divergent series by using some resummation method; divergent series can be convergent in another sense, for example in the Borel sense. Unfortunately, the Borel transformation, which allows one to find the function with the same asymptotic expansion as the divergent series, requires knowledge of the large-order behaviour. Great progress in this direction was made in 1977 by Lipatov who proposed an approximate method of estimating the large-order behaviour applicable in any dimension. The method relies on the expansion of the functional integral representing the investigated quantity in any given order around the non-trivial saddle point. The Lipatov ideas were then developed in field theory by Brézin *et al* (1977) and others. These developments showed the relation between the existence of special classical soliton-type solutions of the equations of motion for imaginary time, the large-order behaviour of the perturbation calculus and the existence of the Borel sum (Zinn-Justin 1981a, b, Marciano and Pagels 1978). For a special class of potentials in zero space dimensions (quantum mechanics) Zinn-Justin (1981a) proposed an equivalent approach.

We shall now rederive the general formula for the kth-order coefficient of the perturbative expansion for the ground state energy. Our final result is the same as that obtained by Zinn-Justin (1981a) or Brézin *et al* (1977); nevertheless the derivation proposed in this paper seems to be the simplest one. The potential $v(q)$ is an entire function of q, with the property

$$v(q)=\tfrac{1}{2}q^2+\mathrm{O}(q^3) \tag{1}$$

and we take it in the form $\lambda^{-2}v(\lambda q)$ to obtain λ expansions. We define the action A as

$$A=\int_0^\beta\left(\tfrac{1}{2}\dot{q}^2+\frac{1}{\lambda^2}v(\lambda q)\right)\mathrm{d}t. \tag{2}$$

The ground state energy is given by the asymptotic expansion

$$E_0(g)=\sum_0^\infty E_k g^k \tag{3}$$

where $g=\lambda^2$ or $g=\lambda$ depending whether $\lambda^{-2}v(\lambda q)$ is a function of λ^2 or of λ.

We shall calculate $E_0(g)$ through

$$E_0(g)=\lim_{\beta\to\infty} -\frac{1}{\beta}\ln\left(\frac{\mathrm{Tr}\,\mathrm{e}^{-\beta H}}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}}\right)+\tfrac{1}{2}. \tag{4}$$

Because the following formula is valid in the semiclassical limit (see Zinn-Justin 1981a)

$$\langle q'|\mathrm{e}^{-\beta H}|q\rangle \sim \frac{1}{(2\pi)^{1/2}}\left(-\frac{\partial^2 A_c}{\partial q(t)\,\partial q(t')}\right)^{1/2}\mathrm{e}^{-A_c}, \qquad \beta=t'-t, \tag{5}$$

we have

$$\begin{aligned}(\mathrm{Tr}\,\mathrm{e}^{-\beta H})_k &= \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}g\int \mathrm{d}q(0)\frac{1}{(2\pi)^{1/2}}\left(-\frac{\partial^2 A_c}{\partial q(0)\,\partial q(\beta)}\right)^{1/2}\frac{\mathrm{e}^{-A_c}}{g^{k+1}}\\ &= Z_k, \qquad q(0)=q(\beta),\end{aligned} \tag{6}$$

with $\mathrm{Tr}\,\mathrm{e}^{-\beta H}=\Sigma_0^\infty Z_k g^k$.

Here the cut is assumed in the complex g plane along the negative or positive real axis and the sum is taken over the periodic orbits. We restrict ourselves to the periodic trajectories because they give the minimal classical action A_c. It is a special case of the general behaviour which we meet in field theory, where one shows with the help of Sobolev inequalities that solutions with minimal action are spherically symmetric (see Zinn-Justin 1981a).

The equation of motion has the form

$$\tfrac{1}{2}\dot r^2(t)=v[r(t)]+E, \qquad r(t)=\lambda q_c(t). \tag{7}$$

Thus as usual the time interval β is equal to

$$\beta=\int_{x'}^{x}\frac{\mathrm{d}r}{[2(v(r)+E)]^{1/2}}, \qquad x=\lambda q(\beta), \qquad x'=\lambda q(0). \tag{8}$$

The classical action is given by

$$A_c=\int_0^\beta (\tfrac{1}{2}\dot q_c^2(t)+\lambda^{-2}v(\lambda q_c))=\frac{1}{g}\left(\int_{x'}^{x}[2(v(r)+E)]^{1/2}\,\mathrm{d}r-E\beta\right). \tag{9}$$

As shown by Zinn-Justin (1981a) the asymptotic form of $E(\beta)$ is ($E\to 0$, $\beta\to\infty$)

$$E(\beta)\sim -\mathrm{e}^{-\beta}2x_+^2\exp\left[2\int_0^{x_+}\left(\frac{1}{(2v(x))^{1/2}}-\frac{1}{x}\right)\mathrm{d}x\right] \tag{10}$$

where x_+ denotes the zero of the potential. In the derivation of (10) the following trick was used:

$$\begin{aligned}\beta &= 2\int_{x_-}^{x_+}\left(\frac{1}{[2(v(r)+E)]^{1/2}}-\frac{1}{(r^2+2E)^{1/2}}+\frac{1}{(r^2+2E)^{1/2}}\right)\mathrm{d}r\\ &\underset{E\to 0}{=} 2\int_0^{x_+}\left(\frac{1}{(2v(r))^{1/2}}-\frac{1}{r}\right)\mathrm{d}r+2\int_0^{x_+}\frac{\mathrm{d}r}{(r^2+2E)^{1/2}}\end{aligned} \tag{11}$$

where x_+, x_- are the turning points of the orbit tending on the basis of equation (7) to the zeros of the potential (in the limit $E \to 0$).

As can be easily shown, the following relations are valid

$$\partial A_c/\partial\beta = -(1/g)E(\beta), \qquad \partial E/\partial\beta = -E, \tag{12}$$

where equations (7), (9) and (10) have been used (or equivalently (2) and (10)). Thus, taking into account (10) and (12), we have

$$A_c = A(\infty)/g + (1/g)E(\beta) \tag{13}$$

where $A(\infty) = 2\int_0^{x_+} (2v(r))^{1/2}\,\mathrm{d}r$.

We can now calculate the second derivative from equation (6). With the help of (13) and (10) we find

$$\frac{\partial A_c}{\partial q(0)} = \frac{1}{g}\frac{\partial E(\beta)}{\partial q(0)} = \frac{1}{g}\frac{\partial E}{\partial\beta}\frac{\partial\beta}{\partial q(0)},$$

$$\frac{\partial^2 A_c}{\partial q(0)\,\partial q(\beta)} = \frac{1}{g}\frac{\partial^2 E}{\partial\beta^2}\frac{\partial\beta}{\partial q(\beta)}\frac{\partial\beta}{\partial q(0)} + \frac{1}{g}\frac{\partial E}{\partial\beta}\frac{\partial^2\beta}{\partial q(0)\,\partial q(\beta)},$$

but

$$\frac{\partial E}{\partial\beta}\frac{\partial^2\beta}{\partial q(0)\,\partial q(\beta)} = -E\frac{(-1)}{[2(v(x')+E)]^{1/2}}\frac{(-1)}{2(v(x')+E)}\frac{\partial E}{\partial\beta}\frac{\partial\beta}{\partial q(\beta)}$$
$$\sim \mathrm{O}(\mathrm{e}^{-2\beta}). \tag{14}$$

So to leading order we obtain

$$\frac{\partial^2 A_c}{\partial q(0)\,\partial q(\beta)} = \frac{\partial E}{\partial\beta}\frac{1}{[2(v(x')+E)]^{1/2}}\frac{1}{[2(v(x)+E)]^{1/2}} \tag{15}$$

where equations (8) and (12) have been used. Finally the coefficient E_k is given by the formula

$$E_{0k} \sim -\sum\frac{1}{2(\pi)^{3/2}}x_+ \exp\int_0^{x_0}\left(\frac{1}{(2v(r))^{1/2}} - \frac{1}{r}\right)\mathrm{d}r\frac{\Gamma(k+1/2)}{A(\infty)^{k+1/2}} \tag{16}$$

valid in the case $\lambda^2 = g$ and where the sum is taken over the non-trivial zeros of the potential.

When $g = \lambda$ the derivation is slightly different. The classical action has the form (see equation (2))

$$A_c = \frac{1}{g^2}\left(\int_{x'}^{x}[2(v(r)+E)]^{1/2}\,\mathrm{d}r - E\beta\right) \tag{17}$$

and in the limit $\beta \to \infty$ we obtain

$$A_c = A(\beta)/g^2 + E(\beta)/g^2. \tag{18}$$

The final formula for the coefficient E_{0k} is

$$E_{0k} \approx -\sum\frac{1}{4\pi^{3/2}}x_+ \exp\int_0^{x_+}\left(\frac{1}{(2v(r))^{1/2}} - \frac{1}{r}\right)\mathrm{d}r\frac{\Gamma(k/2+1/2)}{A^{k/2+1/2}} \tag{19}$$

where the cut was assumed along the whole imaginary g axis for $(\operatorname{Re} A(\infty)<0)$ or along the whole real g axis $(\operatorname{Re} A(\infty)>0)$.

We should mention one weakness of the method which appears when zeros of the potential are degenerate real minima. Usually it is the point at which the potential changes its symmetry. In general the potential is given by

$$v(r)=\prod_{i=1}^{n}(r-x_i).$$

In the neighbourhood of the zero x_+ we have

$$v(r)\approx c(r-x_+)^{\alpha}, \qquad r\to x_+, \tag{20}$$

where c is a certain well defined constant and where $\alpha\geqslant 2$ if x_+ is the degenerate minimum. On the other hand, when we calculate the energy $E(\infty)$ from (10) we have to integrate the function

$$\frac{1}{(2v(r))^{1/2}}-\frac{1}{r}=\frac{1}{r(r-x_+)^{\alpha/2}f(r)}-\frac{1}{r} \tag{21}$$

where $f(r)$ is a certain well defined function. Taking for example $\alpha=2$, the first component in (21) has at $r=x_+$ the same kind of singularity as at $r=0$, which is not cancelled by the second component. Thus the exponent in (10) is infinite. Contrary to the assumption $E\to 0$ for $\beta\to\infty$ which led to (10), E does not vanish. This corresponds to the fact that there do not exist *periodic* zero-energy solutions in this limit and we are forced to apply another approach (see Zinn-Justin 1981a, Harrington 1978). We illustrate these remarks on a simple example introduced by Brézin *et al* (1977); namely, let us consider the potential

$$v(r)=\tfrac{1}{2}r^2-\gamma r^3+\tfrac{1}{2}r^4. \tag{22}$$

This potential is symmetric with respect to $r=\pm\frac{1}{2}$ for $\gamma=\pm 1$. The behaviour of the potential around the zero $x_+=\gamma-(\gamma^2-1)^{1/2}$ may be characterised by the index $\alpha(\gamma)$,

$$|v(r)|\approx c|(r-x_+)|^{\alpha(\gamma)}. \tag{23}$$

We see that

$$\alpha=\begin{cases}1, & |\gamma|\neq 0,\\ 2, & |\gamma|=1,\end{cases}$$

so at the point of the symmetry change of the potential $|\gamma|=1$ the value of the index α increases discontinuously. Note that Brézin *et al* (1977) have shown that the critical value $|\gamma|=1$ characterises the transition point from the Borel summable to the Borel non-summable series for the ground state energy.

The integral in (10) is given by (Brézin *et al* (1977))

$$\int_0^{x_+}\mathrm{d}r\left(\frac{1}{(2v(r))^{1/2}}-\frac{1}{r}\right)=-\ln\{\tfrac{1}{2}(\gamma^2-1)^{1/2}[\gamma-(\gamma^2-1)^{1/2}]\}$$

and is really divergent at the critical point $|\gamma|=1$.

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